JOURNAL OF COMPUTATIONAL PHYSICS 53, 90–99 (1984)

Unsteady Separation

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Received January 28, 1983

A numerical method is presented for calculating the unsteady laminar flow over a circular cylinder started impulsively from rest. The governing boundary-layer equations are solved using the method of series truncation and accurate results have been obtained for larger values of time than any other Eulerian coordinate method. The variation of the vorticity, displacement thickness and displacement velocity are presented. These results, in general, confirm the presence of a singularity at a given time. The value of this time agrees with the Lagrangian coordinate method of Van Dommelen and Shen but does not agree with those of Wang, Telionis and Tsahalis. Further, the results tend to support the structure of the singularity as proposed by van Dommelen and Shen.

INTRODUCTION

In the past few years research on the problem of unsteady boundary-layer separation has been the subject of several papers and colloquia. Some of the "results" presented are contradictory and controversial.

In this paper we shall consider the laminar flow in the boundary layer on a circular cylinder which is impulsively started from rest. It is well known that in the steady-state solution of the boundary-layer equations there is a singularity at an angle of about 104.5° from the front stagnation point of the cylinder, see Terrill [1]. This type of singularity is of the Goldstein–Stewartson type [2–4] and beyond this singularity no steady-state solution exists.

Proudman and Johnson [5] and Robins and Howarth [6] established that at the rear stagnation point the boundary-layer thickness does not approach a steady-state limit at large times but instead grows exponentially with time.

The early numerical calculations of the full time-dependent boundary-layer equations by Belcher *et al.* [7] and Collins and Dennis [8, 9] showed no sign of anything untowards occurring up to non-dimensional times of $\tau = 1$ and 1.25, respectively. Here the time has been non-dimensionalised with respect to a/U. However, a numerical investigation by Telionis and Tsahalis [10] suggested that a singularity develops when $\tau \approx 0.65$ at a station $\theta \approx 140°$ from the front stagnation point of the cylinder. They do, however, state that "the singular behaviour is somewhat confused." Cebeci [11], using Keller's two-point finite-difference method, made a very careful calculation but failed to observe this singularity and found that the

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solution was smooth for $\tau < 1.4$. This naturally lead Cebeci to suggest that the solution of the boundary-layer equations remain smooth for all finite time, even though its thickness is likely to increase exponentially towards the rear of the cylinder.

The complex flow development observed in the experiments of Tietjens [12] and Bouard and Coutanceau [13] suggest that this postulate is too simple and the numerical results of Van Dommelen and Shen [14, 15] confirm this. They use Lagrangian boundary-layer coordinates and numerically "prove" that the spontaneous generation of a singularity does occur at a time $\tau \approx 1.50$, $\theta \approx 111.0°$ and moves upstream with a velocity of 0.52 that of the cylinder. Using an Eulerian finite-difference scheme Wang [16, 17] reports some quantitative differences from Van Dommelen and Shen's results, namely, in the displacement thickness for $\tau > 1$ and he also finds a singularity at $\tau \approx 1.4$.

Van Dommelen and Shen [15] clearly demonstrate in their Fig. 2 the upper limits of the validity of the results obtained by Belcher *et al.* and Cebeci and Wang because of the angular mesh size used in these calculations. Further Van Dommelen and Shen [15] discuss the analytical structure of the singularity and their numerical results tend to confirm this theory. Cowley [18] using a series truncation method in the angular direction and in time with various Padé approximations could obtain accurate results up to $\tau = 1.4$. His results qualitatively confirm the results of Van Dommelen and Shen. Cebeci [19] has extended his earlier work [11] and his calculations tend to confirm some of the results as obtained by Van Dommelen and Shen [15].

One of the purposes of this paper is to use an Eulerian coordinate system and numerically calculate accurately results to a value of τ as large as possible. Because of the difficulties with the angular mesh size described above it was decided to use the series truncation method as described by Collins and Dennis [8, 9].

EQUATIONS

Polar coordinates are used with the origin at the centre of the cylinder of radius a. The cylinder is suddenly started from rest with velocity U in the direction $\theta = 0$ and we work in terms of the dimensionless radial and transverse components of velocity (u, v) obtained by dividing the corresponding dimensionless components by U. If the vorticity is non-dimensionalised with respect to U/a to give ζ and the time with respect to a/U to give τ then the Navier–Stokes equations can be written, in modified polar coordinates (ξ, θ), where $\xi = \ln(r/a)$, in the form (see [9])

$$\exp\{2\xi\} \frac{\partial \zeta}{\partial \tau} + \frac{\partial \psi}{\partial \theta}\frac{\partial \zeta}{\partial \xi} - \frac{\partial \psi}{\partial \xi}\frac{\partial \zeta}{\partial \theta} = \frac{2}{R}\left(\frac{\partial^2 \zeta}{\partial \xi^2} + \frac{\partial^2 \zeta}{\partial \theta^2}\right), \tag{1}$$

$$\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \theta^2} = \exp\{2\xi\}\zeta, \tag{2}$$

where

$$u = \exp(-\xi)\frac{\partial\psi}{\partial\theta}, \qquad v = -\exp(-\xi)\frac{\partial\psi}{\partial\xi}, \tag{3}$$

and $R = 2Ua/\nu$ is the Reynolds number.

Equations (1) and (2) have to be solved subject to the initial and boundary conditions

$$\begin{aligned}
&\psi = \frac{\partial\psi}{\partial\xi} = 0, \qquad \text{on} \quad \xi = 0, \quad \tau \geqslant 0,\\
&\exp\{-\xi\}\frac{\partial\psi}{\partial\theta} \to \cos\theta, \qquad \exp\{-\xi\}\frac{\partial\psi}{\partial\xi} \to \sin\theta \qquad \text{as} \quad \xi \to \infty, \quad \tau \geqslant 0,\\
&\zeta \to 0, \qquad \text{as} \quad \xi \to \infty, \quad \tau \geqslant 0,\\
&\psi = \zeta = 0, \qquad \text{on} \quad \theta = 0^\circ \text{ and } 180^\circ, \quad \xi \geqslant 0, \quad \tau < 0,\\
&\psi = \zeta \equiv 0, \qquad \text{all } \xi \text{ and } \theta, \quad \tau < 0.
\end{aligned} \tag{4}$$

In the early development of the flow a boundary layer of thickness proportional to $(\tau/R)^{1/2}$ forms on the cylinder and therefore the coordinate ξ normal to the cylinder is transformed by

$$\xi = kx, \qquad k = 2(2\tau/R)^{1/2}. \tag{5}$$

This leads to the following scalings for the dependent variables

$$\psi = k\Psi, \qquad \zeta = \omega/k. \tag{6}$$

We shall employ the series truncation method, as described by Collins and Dennis [9], to this problem. Thus we take the following forms of solution in order to satisfy some of the boundary conditions (4):

$$\Psi = \sum_{n=1}^{\infty} F_n(x, \tau)\sin n\theta, \tag{7}$$

$$\omega = \sum_{n=1}^{\infty} G_n(x, \tau)\sin n\theta. \tag{8}$$

In practice the terms in expressions (7) and (8) must be truncated by setting identically zero all terms with a subscript n greater than a prescribed integer, L, say. This defines a truncation of order L. Thus all functions that occur later with a subscript greater than L will be set identically zero.

Substituting expressions (5), (6), (7) and (8) into Eqs. (1) and (2) gives

$$\frac{\partial^2 F_n}{\partial x^2} - n^2k^2F_n = \exp(2kx)\,G_n, \tag{9}$$

$$4\tau \frac{\partial G_n}{\partial \tau} = \exp(-2kx) \frac{\partial^2 G_n}{\partial x^2} + (2x + 4n\tau F_{2n} \exp(-2kx)) \frac{\partial G_n}{\partial x}$$

$$+ \left[2 + \exp(-2kx) \left\{ 2n\tau \frac{\partial F_{2n}}{\partial x} - n^2k^2 \right\} \right] G_n + 4\tau \exp(-2kx)\, S_n, \quad (10)$$

where

$$S_n = \frac{1}{2} \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \left\{ [(m+n) F_{m+n} - jF_j] \frac{\partial G_m}{\partial x} + m \left[\frac{\partial F_{m+n}}{\partial x} - \operatorname{sgn}(m-n) \frac{\partial F_j}{\partial x} \right] G_m \right\}$$

and $j = |m - n|$ and $\operatorname{sgn}(m-n)$ denotes the sign of $m - n$, with $\operatorname{sgn}(0) = 0$.
Using (7) and (8) the boundary conditions (4) become

$$F_n = \frac{\partial F_n}{\partial x} = 0, \qquad \text{on} \quad x = 0, \quad \tau \geqslant 0, \tag{11a}$$

$$k \exp(-kx) F_n \to \delta_n, \qquad \exp(-kx) \frac{\partial F_n}{\partial x} \to \delta_n \qquad \text{as} \quad x \to \infty, \quad \tau \geqslant 0, \tag{11b}$$

where $\delta_1 = 1$, $\delta_n = 0$ $(n = 2, 3, 4, \ldots)$,

$$G_n \to 0, \qquad \text{as} \quad x \to \infty, \quad \tau \geqslant 0. \tag{11c}$$

Collins and Dennis [9] found that the use of an integral condition rather than condition (11b) was easier to implement. Thus if we multiply Eq. (8) by $\exp(-nx)$ and integrate from $x = 0$ to $x = \infty$ we may deduce, using (11a) and (11b), that

$$\int_0^\infty \exp\{(2-n)kx\}\, G_n\, dx = 2\delta_n. \tag{12}$$

Making the usual boundary layer approximation, i.e., $k = 0$, then Eqs. (8) and (9) become

$$\frac{\partial^2 F_n}{\partial x^2} = G_n, \tag{13}$$

$$4\tau \frac{\partial G_n}{\partial \tau} = \frac{\partial^2 G_n}{\partial x^2} + (2x + 4n\tau F_{2n}) \frac{\partial G_n}{\partial x} + \left\{ 2 + 2n\tau \frac{\partial F_{2n}}{\partial x} \right\} G_n + 4\tau S_n, \tag{14}$$

and the boundary conditions (11a), (11c) and (12) become

$$F_n = \frac{\partial F_n}{\partial x} = 0, \qquad \text{on} \quad x = 0, \quad \tau \geqslant 0,$$

$$G_n \to 0, \qquad \text{as} \quad x \to \infty, \quad \tau \geqslant 0, \tag{15}$$

$$\int_0^\infty G_n\, dx = 2\delta_n.$$

Thus the $2L$ equations (13) and (14) have to be solved subject to the boundary conditions (15).

One advantage of the series truncation method is that for small values of τ only a few terms in the series (7) and (8) are required. At $\tau = 0$ only one term is required and as τ increases more and more terms are required in the series.

Equations (13) and (14) were solved in a similar way to that described by Collins and Dennis [9] and hence the details are not presented in full here. Basically Eq. (14) is solved using the Crank–Nicolson method and Eq. (13) as an ordinary differential equation. After each time step a check was performed to see if the order of the truncation should be increased. If so this was done up to $\tau = 1.25$ and the results obtained are indistinguishable from those obtained by Collins and Dennis [9]. However, Collins and Dennis reported that "the integration could not be continued beyond $\tau = 1.25$." If no relaxation parameters are used and a time step of $\Delta\tau = 0.025$, near $\tau = 1.25$, is employed then the present calculations also terminated at $\tau \simeq 1.25$. This is probably due to the appearance of the singularity. By using under-relaxation and much smaller values of $\Delta\tau$ then calculations could be continued to much larger values of τ.

RESULTS

Table I shows the time mesh size used in the calculations. This shows that very small mesh sizes are required for small values of τ and as $\tau \to 1.5$. Several of the time step integrations were performed with smaller values of $\Delta\tau$ in order to check the acuracy but the values as used in Table I were found to give a reasonable accuracy. The mesh size taken in the radial direction was fixed at $h = 0.05$ in all the calculations and the infinity condition in x at 10, x_m, say. It was found that up to the time when the calculations were terminated the choice of x_m was sufficiently large such that all quantities had reached their asymptotic forms.

At $\tau = 0$ only one term in the series truncation is required and as time increases more and more terms are required. Extra terms in the truncation were added if

$$|\max G_L(x, \tau)| > 10^{-5}, \tag{16}$$

then $L = L + 1$ at the next time step for all $\tau < 1.25$. For $\tau > 1.25$ the number of terms required in the series truncation grows rapidly and the criterion (16) has to be

TABLE I

Time Mesh Size Used in Different Time Intervals

Time interval	$0 \to 0.005$	$0.005 \to 0.05$	$0.05 \to 1.25$	$1.25 \to 1.3$	$1.3 \to$
Mesh size	0.0001	0.001	0.025	0.01	0.0025

TABLE II

The Order of Series Truncation at Various Times

Value of τ	0	0.5	1.0	1.25
Value of L	1	19	27	45

abandoned. In Table II the order of truncation at various values of time are presented. Beyond $\tau = 1.25$ it was decided to complete three calculations keeping the value of L fixed. These values of L were 45, 90 and 180. In the case of $L = 45$ calculations were performed up to $\tau = 1.55$, whereas for $L = 90$ and 180 the calculations were terminated at $\tau = 1.47$. The calculations were terminated at these values of τ because of the excessive amount of computing time required. Also the "results" at these values of τ had become too inaccurate because of the order of the series truncation. In all three calculations the numerical method could be integraded through the point $\tau = 1.4$—the point where Wang suggests the equations become singular. Further when $L = 45$ the integrations proceed through the point $\tau = 1.5$—the point where Van Dommelen and Shen suggest that the equations become singular. Figure 1 shows the variation of

$$R^{-1/2}\zeta(0, \theta, \tau) = \frac{1}{2(2\tau)^{1/2}} \sum_{n=1}^{\infty} G_n(0, \tau) \sin n\theta \tag{17}$$

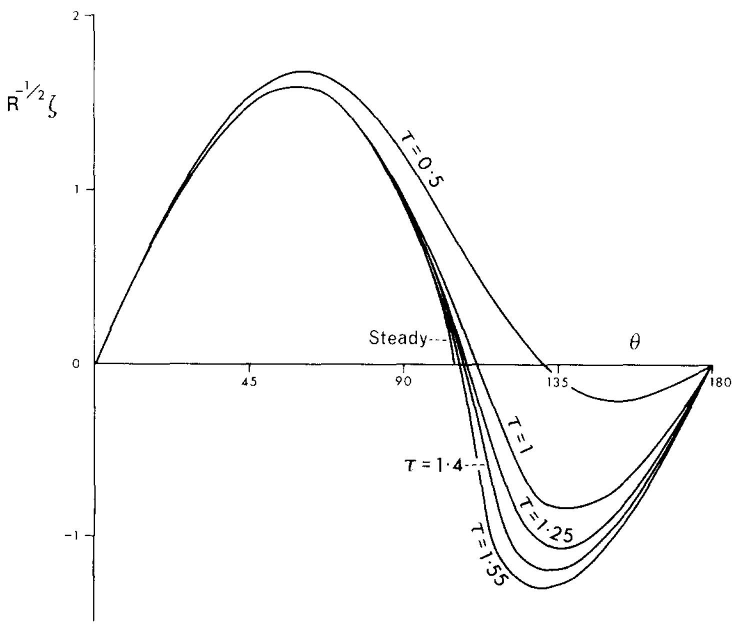

FIG. 1. Variation of $R^{-1/2}\zeta$ over the surface of the cylinder at various values of τ and at steady state.

as a function of the angle θ for various values of τ. This quantity is a measure of $R^{1/2}$ times the local dimensionless coefficient of skin friction on the surface of the cylinder. The results are very similar for the three different orders of truncation up to $\tau = 1.47$ and with $L = 45$ there are no signs of any irregularities up to $\tau = 1.55$. Thus Fig. 1 shows the results with $L = 45$. Also presented is the steady-state boundary-layer solution which has been obtained using a method similar to that described by Terrill [1]. It is seen that this steady boundary-layer solution is being approached as τ increases. (It is well known that the steady-state numerical solution does not describe the observed flows.) The fact that the vorticity does not become singular has been previously observed [14, 15, 18]. This evolution of the wall shear agrees very well with the results of Cebeci [19] up to $\tau = 1.375$.

Van Dommelen and Shen [14, 15] found that the displacement thickness

$$\delta^*(\theta, \tau) = 2(2\tau)^{1/2} \int_0^\infty \left(1 - \frac{\partial \Psi}{\partial x} \Big/ (2 \sin \theta)\right) dx \tag{18}$$

became singular near $\tau = 1.5$. Thus $d\delta^*/d\theta$ will also become singular. In Fig. 2 the position at which δ^* and $d\delta^*/d\theta$ are greatest is plotted as a function of τ. It is seen that the variation of both these quantities is approximately linear with time. Extrapolating the results suggests that a singularity at $\tau \simeq 1.50$ and $\theta \simeq 111.2°$ develops. Further the speed of transmission of the maximum value of δ^* is $0.51\ U$, approximately. These results compare very well with those of Van Dommelen and Shen, namely, $\tau = 1.50$, $\theta = 111.0°$ and speed $0.52\ U$.

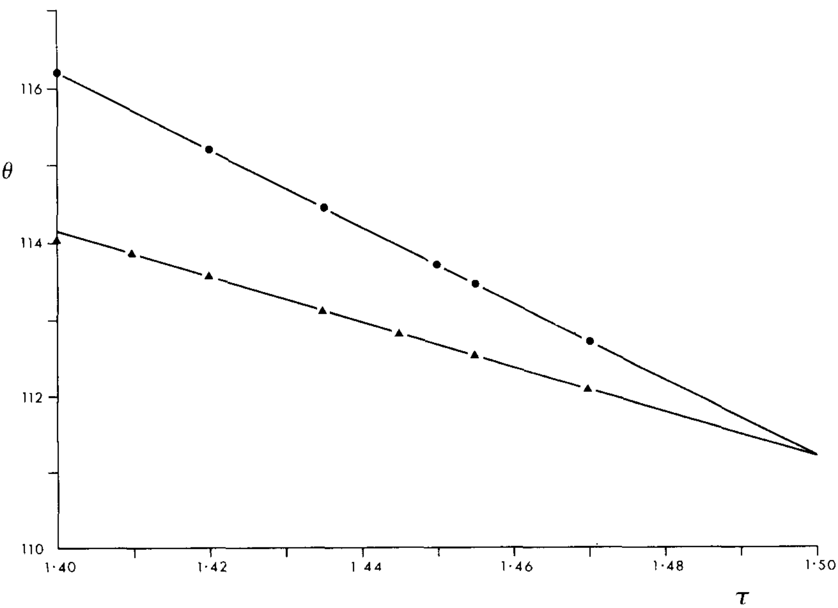

FIG. 2. Variation of the positionof maximum δ^* (●), $\delta\theta^*/\delta\theta$ (▲) and u^* (▲) at various values of τ.

If δ^* becomes singular then the vertical velocity at the outer edge of the boundary layer, which is given by

$$u(x_m, \theta, \tau) = k \sum_{n=1}^{L} nF_n(x_m, \tau) \cos n\theta, \tag{19}$$

will also become singular. Thus we define the viscous displacement velocity u^* by

$$u^*(x_m, \tau, \theta) = R^{1/2}(u(x_m, \tau, \theta) - 2kx \cos \theta). \tag{20}$$

The position of the maximum value of the displacement velocity u^*, $u^*_{\max}$, say, as a function of time is given in Fig. 2. Not surprisingly this position coincides with the position of the maximum value of $d\delta^*/d\theta$.

Assuming that a singularity is being formed at $\tau = 1.50$, we plot in Fig. 3 the variation of $\ln(u^*_{\max})$ with $-\ln(1.5 - \tau)$. It is seen that this is approximately linear with slope 1.75, i.e.,

$$u^*_{\max} \ \alpha \quad (1.5 - \tau)^{-1.75}. \tag{21}$$

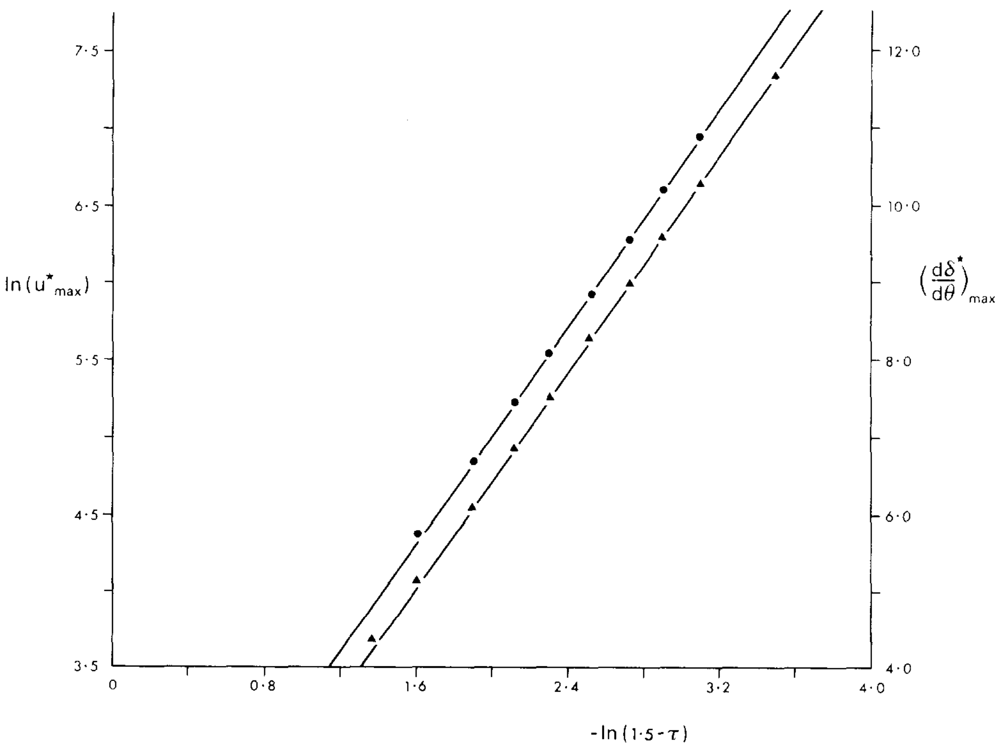

FIG. 3. Variation of $\text{lin}(u^*_{\max})$ (▲) and $\ln((\partial\delta^*/\partial\theta)_{\max})$ (●) as a function of $-\ln(1.5 - \tau)$.

Also in Fig. 3 $(d\delta^*/d\theta)_{max}$ is plotted as a function of $-\ln(1.5-\tau)$ and again it is seen to be approximately linear and that

$$\left(\frac{d\delta^*}{d\theta}\right)_{max} \alpha \quad (1.5-\tau)^{-1.75}. \tag{22}$$

The variations of u^*_{max} and $(d\delta^*/d\theta)_{max}$ as given by expressions (21) and (22) agree with the theory of Van Dommelen and Shen. Other results can be presented which support the theory of the singularity as described by Van Dommelen and Shen, e.g., Van Dommelen and Shen predict that $\delta^*_{max} \alpha \ (1.5-\tau)^{-1/4}$ near $\tau = 1.5$ and the results of the calculations performed here confirm this behaviour for $\tau \gtrsim 1.4$.

It should be noted that Van Dommelen and Shen argue that quantitative verification of their structure requires that $(1.5-\tau)^{1/4}$ should be small, although of course this does not mean that qualitative agreement cannot exist for various quantities as has been obtained in this paper.

Conclusions

Using an Eulerian coordinate system we have illustrated in this paper that it is possible for a singularity to occur at a finite time for the unsteady boundary-layer equations. The predicted values of time and position agree with those predicted by Van Dommelen and Shen (who used a Lagrangian coordinate system) rather than those of Wang [16, 17] and Telionis and Tsahalis [10] (who used an Eulerian coordinate system). The small discrepancy in the position of the singularity, 111.2° compared with 111.0° by Van Dommelen and Shen, could be due to the use of a mesh size in the radial direction which is too crude—because of the large amount of computing time being used it was impossible to perform calculations with an appreciably smaller mesh size in the radial direction.

Further, the numerical results presented here tend to confirm that the nature of the singularity is as described by Van Dommelen and Shen.

Acknowledgments

I wish to thank Professor K. Stewartson and Drs. S. Cowley, F. T. Smith and L. L. Van Dommelen for their helpful comments on the work.

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